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Short Communication

Maximizing the natural frequency of a beam with an intermediate elastic support

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Abstract

The minimum stiffness of a simple (or point) support that raises a natural frequency of a beam to its upper limit is investigated for different boundary conditions. The approach produces the closed-form solution for the minimum stiffness based on the derivative of a natural frequency with respect to the support position. It is seen that when an intermediate elastic support is positioned properly the effect is similar to a rigid support. The solution process also provides insight into the dynamics of a beam with an intermediate support for more general boundary conditions. The resulting solutions can be used to guide the practical design of a support.

1. Introduction

The design of structural supports plays a key role in engineering dynamics and therefore close attention should be paid to their characteristics. Supports are not only expected to hold a structure firmly, but can also be redesigned to improve the structural performance. Thus far, numerous papers have been devoted to this problem [1–7]. Akesson and Olhoff [2] pointed out that there exists a certain minimum stiffness of an additional support when maximizing the

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fundamental frequency of a cantilever beam. Increasing the support stiffness over the minimum value cannot raise the lowest frequency further due to mode switching. Rao [3] developed the frequency equation of a beam with an intermediate support by the continuity conditions at the supported point. Wang [4] numerically evaluated the minimum stiffness of an elastic support for beams with different end conditions. Wang et al. [5] derived the frequency derivative of a beam-like structure with regard to the position of a simple (or point) support by the discrete method. Moreover, they presented a procedure to determine the optimum positions of elastic supports based on the frequency sensitivity. Albarracín et al. [6] calculated the effect of an intermediate support when the ends of the beam have elastic constraints. Low [8,9] considered the effect of a discrete mass on the beam natural frequencies for a variety of end conditions.

The minimum stiffness of an additional support required to maximize a natural frequency is of particular interest in engineering applications since producing a support with infinite stiffness is virtually impossible. Thus, designing an elastic support at the optimum position that gives a similar effect to a rigid support has significant advantages. So far, no explicit solution has been derived to calculate the minimum support stiffness, although numerical solutions are possible when the support is placed at a node of a higher mode [4]. This study derives the closed-form solution for the minimum stiffness using the derivatives of a natural frequency with respect to the support position. The solution process also provides insight into the dynamics of a beam with an intermediate support for more general boundary conditions. Furthermore, the present procedure is easily extended to the higher natural frequencies.

2. Dynamics of a vibrating beam

Fig. 1 shows a uniform cantilever Euler–Bernoulli beam with flexural rigidity EI, mass per unit length m and length L. For convenience the axial location of a cross section is indicated with a dimensionless coordinate x. Assume an elastic support with stiffness k is located at $x = b$. The eigenvalue equation for the vibration of the beam is

$$w''''(x) - \lambda^4 w(x) = 0, \tag{1}$$

where

$$\lambda^4 = \frac{\omega^2 m L^4}{EI} \tag{2}$$

and $w(x)$ is the transverse displacement of the beam, ω the natural (circular) frequency of vibration and λ the frequency parameter. The prime denotes differentiation with respect to x. The

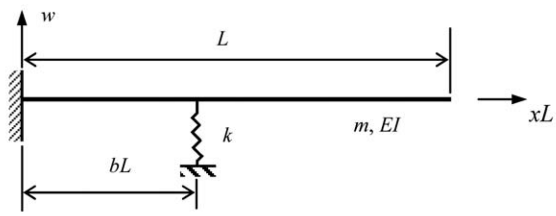

Fig. 1. A uniform cantilever beam with an intermediate support.

general solution to Eq. (1) for the cantilever beam is defined separately over those parts of the beam either side of the support location, denoted by $w_1(x)$ and $w_2(x)$. For each solution the general form of the displacement given by Eq. (1) has four unknown parameters. Applying the clamped boundary condition to $w_1(x)$ and the free boundary condition to $w_2(x)$ gives the following general solution:

$$w_1(x) = C_1[\sinh \lambda x - \sin \lambda x] + C_2[\cosh \lambda x - \cos \lambda x] \quad \text{for} \quad 0 \leqslant x \leqslant b, \tag{3}$$

$$\begin{aligned} w_2(x) = {} & C_3[\sinh \lambda(x-1) + \sin \lambda(x-1)] \\ & + C_4[\cosh \lambda(x-1) + \cos \lambda(x-1)] \quad \text{for} \quad b \leqslant x \leqslant 1. \end{aligned} \tag{4}$$

Wang [4] presented detailed solutions for other end conditions. The compatibility conditions for the deflections and internal forces at the support position will determine the constants C_i in Eqs. (3) and (4), and also the natural frequencies of the beam [3].

According the Courant's maximum–minimum principle [1], an additional support can increase the structural frequency, ω_i, to between the ith and the (i+1)th natural frequencies of the original system. Akesson and Olhoff [2] showed that if a support with the minimum stiffness, k_0, is located at the node of the original second mode, and that the first natural frequency of the beam system will reach its upper limit value of the second natural frequency of the beam without the support. Wang et al. [5] showed that the derivative of the ith natural frequency with respect to the support position is

$$\frac{\partial \omega_i^2}{\partial b} = 2Lkw_i(b)\theta_i(b), \tag{5}$$

where $w_i(b)$ and $\theta_i(b)$ denote the transverse displacement and the slope, respectively, of the ith vibration mode at the support. Hence, to maximize the first natural frequency, either the displacement or the slope of the first vibration mode must vanish at the support position. If the support with the minimum stiffness, k_0, is located at the node of the second mode of the original system, then the displacement of the first mode will be non-zero and hence the slope of the first mode must vanish.

3. Calculating the minimum stiffness

Once the support stiffness is given, Eq. (5) enables us to find both the optimum support position and the corresponding maximum frequency [5]. Alternatively, if an appropriate support position, say b, is prescribed as the optimum, which often arises in practice, we can also determine the support stiffness, k_0, required and the corresponding maximum frequency parameter, λ. Moreover, it should be stressed that this stiffness is the minimum value to obtain the frequency parameter λ. In other words, it is impossible to obtain the frequency λ with a lower support stiffness. To fix the idea, we consider two typical sets of end conditions for the beam.

3.1. Cantilever beam

A cantilever beam with an elastic support is shown in Fig. 1. Suppose that the support is located at b and makes the slope of the first mode equal to zero at b. Then, from Eqs. (3) and (4), we have

$$w_1'(b) = C_1\lambda[\cosh \lambda b - \cos \lambda b] + C_2\lambda[\sinh \lambda b + \sin \lambda b] = 0, \tag{6a}$$

$$w_2'(b) = C_3\lambda[\cosh\lambda(b-1) + \cos\lambda(b-1)] + C_4\lambda[\sinh\lambda(b-1) + \sin\lambda(b-1)] = 0 \quad (6b)$$

and thus

$$C_2 = -C_1 \frac{\cosh\lambda b - \cos\lambda b}{\sinh\lambda b + \sin\lambda b} \quad \text{and} \quad C_4 = -C_3 \frac{\cosh\lambda(b-1) + \cos\lambda(b-1)}{\sinh\lambda(b-1) - \sin\lambda(b-1)}. \quad (7)$$

The continuity conditions of the displacement and the bending moment (the second derivative of the displacement) at the support position yield,

$$C_1 \frac{2\cosh\lambda b\cos\lambda b - 2}{\sinh\lambda b + \sin\lambda b} = -C_3 \frac{2\cosh\lambda(b-1)\cos\lambda(b-1) + 2}{\sinh\lambda(b-1) - \sin\lambda(b-1)}, \quad (8a)$$

$$C_1 \frac{2\sinh\lambda b\sin\lambda b}{\sinh\lambda b + \sin\lambda b} = -C_3 \frac{2\sinh\lambda(b-1)\sin\lambda(b-1)}{\sinh\lambda(b-1) - \sin\lambda(b-1)}. \quad (8b)$$

The requirement for non-trivial solutions for C_1 and C_3 in Eqs. (8) produces the characteristic determinant equation for the maximum fundamental frequency parameter λ as

$$\begin{vmatrix} \cosh\lambda b\cos\lambda b - 1 & \cosh\lambda(b-1)\cos\lambda(b-1) + 1 \\ \sinh\lambda b\sin\lambda b & \sinh\lambda(b-1)\sin\lambda(b-1) \end{vmatrix} = 0. \quad (9)$$

Only three of the four continuity conditions have been used to calculate the maximum frequency parameter. The continuity of the shear forces at the supported location is used to determine the minimum support stiffness, k_0, as [4]

$$w_1'''(b) - \gamma w_1(b) = w_2'''(b), \quad (10)$$

where $\gamma = k_0L^3/EI$ is the normalized or dimensionless minimum stiffness. It follows immediately that

$$\gamma = \frac{w_1'''(b) - w_2'''(b)}{w_1(b)}. \quad (11)$$

For example, positioning a support at the node of the second mode of the cantilever beam, i.e., $b = 0.7834$, the solution to Eq. (9) is $\lambda = 4.6941$. That is, the first frequency of the supported beam is equal to the second natural frequency of the beam without the additional support and then a repeated fundamental natural frequency occurs in the system. The minimum support stiffness is obtained from Eq. (11) as $\gamma = 266.87$. Over that value the first natural frequency cannot be raised further by increasing the support stiffness, whereas the second natural frequency would be raised. As a second example suppose that $b = 1.0$, and then Eq. (9) reduces to

$$\sinh\lambda\sin\lambda = 0. \quad (12)$$

The smallest non-zero solution is

$$\lambda = \pi \quad (13a)$$

and thus

$$\gamma = 28.44. \quad (13b)$$

Figs. 2 and 3 plot the minimum support stiffness and the maximum fundamental frequency, respectively, versus the support position. Only support positions from the node of the second

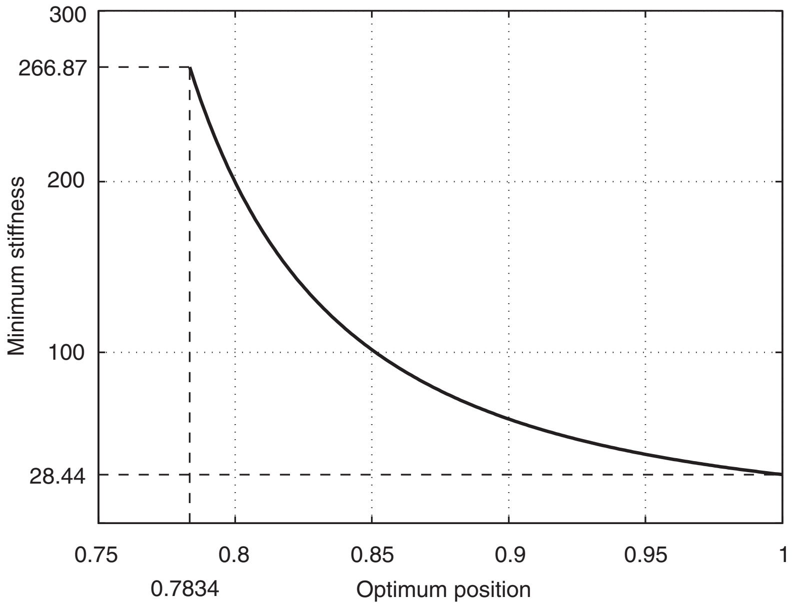

Fig. 2. Minimum support stiffness required for a given support position.

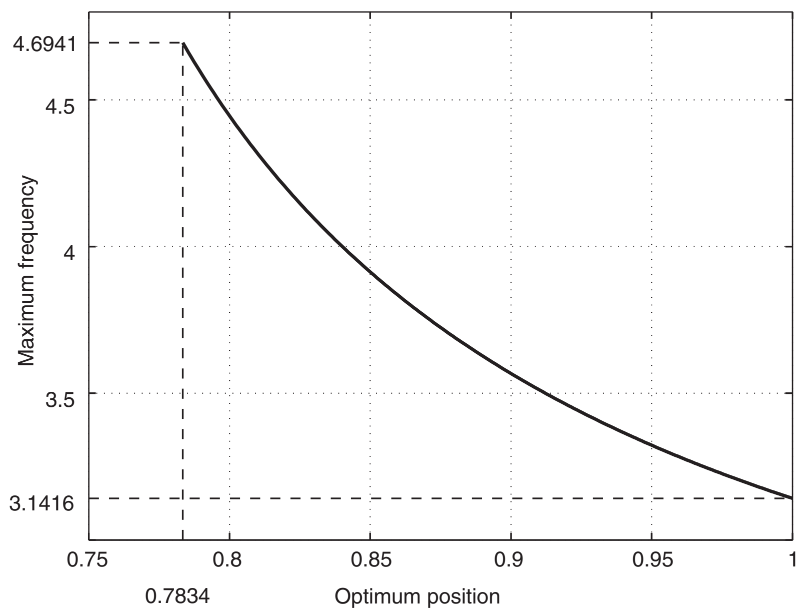

Fig. 3. Maximum fundamental frequency of a cantilever beam achieved for a given support position with the minimum stiffness.

mode through to the beam tip are included, since no optimum exists for locations closer to the clamped end [5]. It is observed that the beam's fundamental frequency may be changed significantly by moving the support. When $b = 0.8278$ the minimum stiffness is $\gamma = 133.37$, and the maximum frequency is $\lambda = 4.1186$, as given by Akesson and Olhoff [2]. Note that if the support stiffness is above 266.87, the optimum position remains at the node of the second mode of the unsupported beam, whereas if the stiffness is below 28.44, the optimum position will be at the beam tip [5].

3.2. Simply supported beam

We now consider the case of symmetric end conditions. For a simply supported beam with an intermediate support the beam displacement may also be written in a similar manner to Eqs. (3) and (4), where the displacement functions now satisfy the pinned boundary conditions [4]. The characteristic determinant equation for the frequency parameter is obtained by considering continuity at the support and the zero slope condition, in a similar way to the clamped beam example. The resulting equation is

$$\begin{vmatrix} \sinh\lambda b\cos\lambda b - \cosh\lambda b\sin\lambda b & \sinh\lambda(b-1)\cos\lambda(b-1) - \cosh\lambda(b-1)\sin\lambda(b-1) \\ \sinh\lambda b\cos\lambda b + \cosh\lambda b\sin\lambda b & \sinh\lambda(b-1)\cos\lambda(b-1) + \cosh\lambda(b-1)\sin\lambda(b-1) \end{vmatrix} = 0. \quad (14)$$

If $b = 1/2$ [4], then the above equation becomes

$$\begin{vmatrix} \sinh\frac{\lambda}{2}\cos\frac{\lambda}{2} - \sin\frac{\lambda}{2}\cosh\frac{\lambda}{2} & \sinh\frac{\lambda}{2}\cos\frac{\lambda}{2} - \sin\frac{\lambda}{2}\cosh\frac{\lambda}{2} \\ \sinh\frac{\lambda}{2}\cos\frac{\lambda}{2} + \sin\frac{\lambda}{2}\cosh\frac{\lambda}{2} & \sinh\frac{\lambda}{2}\cos\frac{\lambda}{2} + \sin\frac{\lambda}{2}\cosh\frac{\lambda}{2} \end{vmatrix} = 0. \quad (15)$$

This equation is satisfied for all values of λ, which implies that for any support stiffness the optimum position is always at the mid-span of the beam. Thus the beam center is the only optimum support position for the simply supported beam, which is obvious for beams with symmetric end conditions. In this case, the first natural frequency is then a function of the deflection of the beam at the support location. If the first natural frequency is increased to that for the second mode of the unsupported structure, then $\lambda = 2\pi$ [4], and

$$\gamma = \frac{1}{w_1(1/2)}[w_1'''(1/2) - w_2'''(1/2)] = 32\pi^3\frac{\cosh(\pi)}{\sinh(\pi)} = 995.91. \quad (16)$$

Suppose that the support is at position $b = 0.75$, then a solution to Eq. (14) is $\lambda = 2\pi$. However, the required stiffness of the support is then computed to be $\gamma = 0$, which is unreasonable.

3.3. Other end conditions

The lowest natural frequency of a uniform beam may be maximized for different sets of end conditions. The resulting characteristic equations and the minimum support stiffnesses are tabulated in the Appendix. The free–free case is not included since such a beam with a point support will still have a rigid body mode. Also shown are some special cases where the support is located at the node of the second mode of the unsupported beam to verify the numerical results of Wang [4].

4. Minimum stiffness for higher natural frequencies

In addition to raising the first natural frequency of a beam to its upper limit with an elastic support, we can also increase any of the higher natural frequencies to its upper limit with a flexible support. The optimum position for this support is at the nodes of the next higher frequency modes and the minimum stiffness may be evaluated using the method described previously. For example, placing the lateral support at $b = 0.8677$, which is one of the nodes of the third mode of the cantilever beam [2], the second smallest solution to Eq. (9) is $\lambda = 7.8548$, which is the third natural frequency of the unsupported structure. Consequently, Eq. (11) may be used to calculate the minimum support stiffness as $\gamma = 1307.5$ and this support stiffness will raise the second natural frequency to equal the third. If the support is placed at another node of the third mode shape, $b = 0.5035$, then we obtain solutions $\lambda = 7.8548$ and $\gamma = 1942.5$. It is seen that the required minimum stiffness is quite different at each of the nodes of the third mode. For the simply supported beam, taking $b = 1/3$ or $2/3$, the solution to Eq. (14) is $\lambda = 3\pi$ and then Eq. (11) is $\gamma = 3354.9$, for both positions. Tables 1 and 2 give the optimum position and the minimum

Table 1
Optimal support position, minimum stiffness and maximum frequency for raising the higher natural frequencies of a cantilever beam

Mode no.	Original frequency (λ^i)	Optimum position (b)	Maximum frequency (λ^u)	Minimum stiffness (γ)	Stiffness at other mode nodes
1	1.8751	0.7834	4.6941	266.87	
2	4.6941	0.8677	7.8548	1307.5	$\gamma_{(0.5035)} = 1942.5$
3	7.8548	0.9055	10.996	3584.1	$\gamma_{(0.6440)} = 5487.3$
					$\gamma_{(0.3583)} = 5164.3$
4	10.996	0.9265	14.137	7612.5	$\gamma_{(0.7232)} = 11631$
					$\gamma_{(0.4999)} = 11301$
					$\gamma_{(0.2788)} = 10999$

Table 2
Optimal support position, minimum stiffness and maximum frequency for raising the higher natural frequencies of a simply supported beam

Mode no.	Original frequency (λ^i)	Optimum position (b)	Maximum frequency (λ^u)	Minimum stiffness (γ)	Stiffness at other mode nodes
1	3.1416	0.5000	6.2832	995.91	
2	6.2832	0.3333	9.4248	3354.9	
		0.6667			
3	9.4248	0.5000	12.566	7937.7	$\gamma_{(0.25)} = 7952.5$
					$\gamma_{(0.75)} = 7952.5$
4	12.566	0.4000	15.708	15503	$\gamma_{(0.2)} = 15532$
		0.6000			$\gamma_{(0.8)} = 15532$

stiffness of a support to increase the first four natural frequencies for cantilever and simply supported beams, respectively. In each case the natural frequency of the mode of interest is raised to equal the next higher natural frequency. The results show that the minimum support stiffness required increases rapidly for the higher natural frequencies.

5. Conclusions

Once the optimum position of an additional support is prescribed properly, a natural frequency of a beam can be raised to its upper limit with a minimum requirement on the support stiffness. By calculating the natural frequency derivatives, the problem of finding the minimum stiffness reduces to determination of the zero slope of the related mode shape. In this study, the analytical formulation of the minimum support stiffness is developed for different types of beam-end conditions in a systematic manner based on the continuity conditions at the support point. Furthermore, it is illustrated that a stiffer intermediate support is needed to raise a higher order frequency to its maximum. The resulting solutions can be used in the design of practical supports.

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Appendix

The characteristic determinant equation and the minimum support stiffness for various sets of beam end conditions are given in Table 3.

Table 3

Ends	Characteristic determinant equation (= 0)	Minimum support stiffness $\left(\gamma = \frac{k_0 L^3}{EI}\right)$	b	λ	γ
C–C	$\begin{vmatrix} \cosh\lambda b\cos\lambda b-1 & \cosh\lambda(b-1)\cos\lambda(b-1)-1 \\ \sinh\lambda b\sin\lambda b & \sinh\lambda(b-1)\sin\lambda(b-1) \end{vmatrix}$	$2\lambda^3\frac{\sinh\lambda/2\cos\lambda/2+\cosh\lambda/2\sin\lambda/2}{\cosh\lambda/2\cos\lambda/2-1}$	$\frac{1}{2}$	7.88532	1833.66
C–S	$\begin{vmatrix} \cosh\lambda b\cos\lambda b-1 & \sinh\lambda(b-1)\cos\lambda(b-1)-\cosh\lambda(b-1)\sin\lambda(b-1) \\ \sinh\lambda b\sin\lambda b & \sinh\lambda(b-1)\cos\lambda(b-1)+\cosh\lambda(b-1)\sin\lambda(b-1) \end{vmatrix}$	$\lambda^3\left[\frac{\sinh\lambda b\cos\lambda b+\cosh\lambda b\sin\lambda b}{\cosh\lambda b\cos\lambda b-1}-\frac{2\cos\lambda(b-1)\cosh\lambda(b-1)}{\sinh\lambda(b-1)\cos\lambda(b-1)-\cosh\lambda(b-1)\sin\lambda(b-1)}\right]$	0.5575	7.0681	1377.60
C–Sl	$\begin{vmatrix} \cosh\lambda b\cos\lambda b-1 & \cosh\lambda(b-1)\sin\lambda(b-1)+\sinh\lambda(b-1)\cos\lambda(b-1) \\ \sinh\lambda b\sin\lambda b & \cosh\lambda(b-1)\sin\lambda(b-1)-\sinh\lambda(b-1)\cos\lambda(b-1) \end{vmatrix}$	$\lambda^3\left[\frac{\sinh\lambda b\cos\lambda b+\cosh\lambda b\sin\lambda b}{\cosh\lambda b\cos\lambda b-1}-\frac{2\sinh\lambda(b-1)\sin\lambda(b-1)}{\cosh\lambda(b-1)\sin\lambda(b-1)+\sinh\lambda(b-1)\cos\lambda(b-1)}\right]$	0.7169	5.4977	619.39
C–F	$\begin{vmatrix} \cosh\lambda b\cos\lambda b-1 & \cosh\lambda(b-1)\cos\lambda(b-1)+1 \\ \sinh\lambda b\sin\lambda b & \sinh\lambda(b-1)\sin\lambda(b-1) \end{vmatrix}$	$\lambda^3\left[\frac{\sinh\lambda b\cos\lambda b+\cosh\lambda b\sin\lambda b}{\cosh\lambda b\cos\lambda b-1}-\frac{\sinh\lambda(b-1)\cos\lambda(b-1)+\cosh\lambda(b-1)\sin\lambda(b-1)}{\cosh\lambda(b-1)\cos\lambda(b-1)+1}\right]$	0.7834	4.6941	266.87
S–S	$\begin{vmatrix} \sinh\lambda b\cos\lambda b-\cosh\lambda b\sin\lambda b & \sinh\lambda(b-1)\cos\lambda(b-1)-\cosh\lambda(b-1)\sin\lambda(b-1) \\ \sinh\lambda b\cos\lambda b+\cosh\lambda b\sin\lambda b & \sinh\lambda(b-1)\cos\lambda(b-1)+\cosh\lambda(b-1)\sin\lambda(b-1) \end{vmatrix}$	$4\lambda^3\frac{\cosh\lambda/2\cos\lambda/2}{\sinh\lambda/2\cos\lambda/2-\cosh\lambda/2\sin\lambda/2}$	$\frac{1}{2}$	2π	995.91
S–Sl	$\begin{vmatrix} \sinh\lambda b\cos\lambda b-\cosh\lambda b\sin\lambda b & \cosh\lambda(b-1)\sin\lambda(b-1)+\sinh\lambda(b-1)\cos\lambda(b-1) \\ \sinh\lambda b\cos\lambda b+\cosh\lambda b\sin\lambda b & \cosh\lambda(b-1)\sin\lambda(b-1)-\sinh\lambda(b-1)\cos\lambda(b-1) \end{vmatrix}$	$2\lambda^3\left[\frac{\cosh\lambda b\cos\lambda b}{\sinh\lambda b\cos\lambda b-\cosh\lambda b\sin\lambda b}-\frac{\sinh\lambda(b-1)\sin\lambda(b-1)}{\cosh\lambda(b-1)\sin\lambda(b-1)+\sinh\lambda(b-1)\cos\lambda(b-1)}\right]$	$\frac{2}{3}$	$\frac{3\pi}{2}$	402.03
S–F	$\begin{vmatrix} \sinh\lambda b\cos\lambda b-\cosh\lambda b\sin\lambda b & \cosh\lambda(b-1)\cos\lambda(b-1)+1 \\ \sinh\lambda b\cos\lambda b+\cosh\lambda b\sin\lambda b & \sinh\lambda(b-1)\sin\lambda(b-1) \end{vmatrix}$	$\lambda^3\left[\frac{2\cosh\lambda b\cos\lambda b}{\sinh\lambda b\cos\lambda b-\cosh\lambda b\sin\lambda b}-\frac{\sinh\lambda(b-1)\cos\lambda(b-1)+\cosh\lambda(b-1)\sin\lambda(b-1)}{\cosh\lambda(b-1)\cos\lambda(b-1)+1}\right]$	0.7358	3.9263	163.55
Sl–Sl	$\begin{vmatrix} \cosh\lambda b\sin\lambda b+\sinh\lambda b\cos\lambda b & \cosh\lambda(b-1)\sin\lambda(b-1)+\sinh\lambda(b-1)\cos\lambda(b-1) \\ \cosh\lambda b\sin\lambda b-\sinh\lambda b\cos\lambda b & \cosh\lambda(b-1)\sin\lambda(b-1)-\sinh\lambda(b-1)\cos\lambda(b-1) \end{vmatrix}$	$4\lambda^3\frac{\sinh\lambda/2\sin\lambda/2}{\sinh\lambda/2\cos\lambda/2+\cosh\lambda/2\sin\lambda/2}$	$\frac{1}{2}$	π	113.75
Sl–F	$\begin{vmatrix} \cosh\lambda b\sin\lambda b+\sinh\lambda b\cos\lambda b & \cosh\lambda(b-1)\cos\lambda(b-1)+1 \\ \cosh\lambda b\sin\lambda b-\sinh\lambda b\cos\lambda b & \sinh\lambda(b-1)\sin\lambda(b-1) \end{vmatrix}$	$\lambda^3\left[\frac{2\sinh\lambda b\sin\lambda b}{\sinh\lambda b\cos\lambda b+\cosh\lambda b\sin\lambda b}-\frac{\sinh\lambda(b-1)\cos\lambda(b-1)+\cosh\lambda(b-1)\sin\lambda(b-1)}{\cosh\lambda(b-1)\cos\lambda(b-1)+1}\right]$	0. 5517	2.3731	33.491

The notation for the end conditions is C = clamped, S = simply supported, Sl = sliding, F = free.

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